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THE SLOW ROTATION OF A SPHERE SUBMERGED IN A FLUID WITH A SURFACTANT SURFACE LAYER

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Abstract—The effect of a layer of an adsorbed surfactant monomolecular film of fluid which covers the surface of a large volume of a different substrate fluid is considered with respect to the fluid motion caused by the slow rotation of a submerged sphere. For a semi-infinite substrate, the boundary value problem posed with the surfactant boundary condition of Scriven and Goodrich is solved exactly for any depth of the submerged sphere. Comprehensive numerical calculations are given for the torque and surface velocity for various values of the parameters defining the depth of the sphere and the surface shear viscosity. Asymptotic expressions for the solution are given for the cases of a deeply submerged sphere or when the substrate has a finite depth. The relevance of the work to providing an experimental technique for measuring surface shear viscosity is also considered.

1. INTRODUCTION

A significant problem of interest in surface chemistry is the effect of an adsorbed surfactant monomolecular film of fluid which covers the surface of a large volume of different fluid—the substrate—and a body moving near to or at the surfactant layer. The presence of the surfactant layer results in the existence of a coefficient of surface shear viscosity κ and the appropriate boundary conditions to apply at the surface, reflecting the effect of the surfactant layer, can be derived from the work of Scriven (1960), who studied the motion of a thin fluid interface between two bulk fluids of different viscosities.

Theoretical and experimental work to measure the coefficient κ has been carried out by Goodrich *et al.* (1969, 1970, 1971). They examined the dynamics of a viscometer for measuring κ which consisted of a thin circular disc inserted into the plane interface between the surfactant film and the underlying substrate. The disc was rotated slowly and the torque required to maintain the steady motion was measured. From a knowledge of this measured torque together with an analytical formula relating the torque to the shear viscosity, the value of κ could accordingly be deduced. Goodrich's theoretical analysis assumes that the Reynolds numbers for the motions of both the surfacant and substrate are sufficiently small for the linearized Stokes equations to apply. The only non-vanishing component of the fluid velocity $\mathbf{v}$ is in the azimuthal direction, and a somewhat unusual mixed boundary value problem results for $|\mathbf{v}|$ due to the boundary condition to be satisfied at the surface expressing the balance of substrate stresses on the adsorbed film and the internal film stresses. Taking (ρ, θ, z) to be cylindrical polar coordinates with the z-axis drawn into the substrate and the plane $z = 0$ within the interface between the surfactant layer and the substrate, this boundary condition takes the form

$$\mu\frac{\partial v}{\partial z} - \kappa\frac{\partial^2 v}{\partial z^2} = 0,$$

where $v = |\mathbf{v}|$ and μ, κ denote respectively the coefficient of internal viscosity of the substrate and surface shear viscosity of the adsorbed film.

The mathematical analysis of Goodrich is not entirely satisfactory, and its shortcomings are discussed in detail by Shail (1978) who has produced a simpler form of solution using the methods of Generalised Axially Symmetric Potential Theory to formulate an integral equation problem for v. Furthermore, Shail's analysis, besides providing a complete set of numerical data for the torque when $\lambda = \kappa/\mu$ takes general values, gives a comprehensive view of the asymptotic structure of the solution for very large and very small values of λ.

The great difficulty encountered in using a rotating disc within a surfactant layer to make measurements of the torque is that of precise positioning of the disc within the layer of adsorbed film. The results of the theoretical work also indicate that this is not a particularly sensitive method for measuring small coefficients of shear viscosity, which magnifies the errors associated with positioning of the disc and casts further doubts on the ability of this type of measurement. This has led the authors in this paper, together with Shail (1979) in a companion paper, to propose an alternative approach to the measurement of κ. By removing the measuring device from actually within the surfactant layer, the difficulty referred to above can be conveniently overcome. Shail has proposed a measuring device based on a rotating disc which is placed in the substrate fluid below the surfactant layer.

In this paper, we propose using a sphere, again suspended in the substrate below the surfactant layer, which rotates slowly about a diameter perpendicular to the plane of the surfactant layer. We feel that the choice of a spherical body is particularly advantageous for this type of experiment in which extremely accurate measurements must be made in order to arrive at a reliable value of κ. All physical parameters can be closely monitored and the inherent disadvantage of a real disc having thickness, unlike a mathematical disc, does not arise, thus excluding a further source of error. Furthermore, this geometry ensures that an exact mathematical solution of the boundary value problem can be found for all depths of the sphere below the surfactant layer, even when the sphere approaches tangency with the layer. We have calculated the values of the torque acting on the sphere for a wide range of values of the depth of the sphere for values of λ extending from zero to infinity. The limiting cases $\lambda = 0$, when the shear viscosity is zero and the surfactant layer becomes a simple stress free surface, and $\lambda = \infty$, when the shear viscosity is infinite corresponding to a solid plane boundary, are also considered. We also give a comprehensive asymptotic analysis of the form of the velocity and torque when $\lambda/h \ll 1$ and $\lambda/h \gg 1$, where h is the depth of the sphere centre below the surfactant layer, for a substrate of infinite depth. The effect of a substrate of finite depth is also considered. An alternative experimental procedure to measuring the torque acting on the sphere would be to measure the surface velocity of the surfactant which is induced by the rotation of the sphere. This can be conveniently achieved by depositing marker particles on the surface and measuring their velocities. We have therefore given numerical data which predicts what the surface velocity distribution would be for varying depths of the sphere and the viscosity ratio parameter λ.

2. SPHERE ROTATING BELOW THE SURFACTANT LAYER

A rigid sphere of radius a rotates in a semi-infinite incompressible fluid with dynamic viscosity μ. The axis of rotation is the diameter of the sphere perpendicular to the upper bounding surface of the fluid on which there is a layer of an adsorbed surfactant monomolecular film possessing surface viscosity κ. The depth of the centre of the sphere below this surfactant layer is h and the sphere rotates with constant angular velocity Ω.

We shall assume that the Reynolds number for the flow induced in both the surfactant layer and the substrate fluid is sufficiently small to permit the neglect of inertia terms in the Navier–Stokes equations. Consequently, the equations governing the flow are

$$\nabla p = \mu \nabla^2 \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0, \tag{2.1}$$

where $\mathbf{v}$ and p are respectively the velocity and pressure of the fluid.

The fluid motion is caused solely by the rotation of the sphere, and because of the axially symmetric nature of the problem, it seems reasonable to suppose that the velocity $\mathbf{v}$ has only one component which is in the azimuthal θ direction of a system of cylindrical polar coordinates (ρ, θ, z) with the z-axis along the axis of rotation of the sphere and pointing into the substrate fluid. The plane $z = 0$ coincides with that of the surfactant layer. Accordingly, [2.1] possess a solution of the form

$$\mathbf{v} = v(\rho, z)\boldsymbol{\theta}, \quad p = 0, \qquad [2.2]$$

provided that

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{1}{\rho^2} + \frac{\partial^2}{\partial z^2}\right)v = 0. \qquad [2.3]$$

The boundary condition on the sphere requires that

$$v = \Omega \rho, \qquad [2.4]$$

for points (ρ, θ, z) on the sphere, and following the work of Scriven (1960), the boundary condition at the surfactant layer, which expresses the balance of substrate stresses on the adsorbed film and the inertial film stresses, takes the form

$$\frac{\partial v}{\partial z} - \lambda \frac{\partial^2 v}{\partial z^2} = 0 \text{ on } z = 0, \qquad [2.5]$$

where $\lambda = \kappa/\mu$. A further kinematic condition requires that $v \to 0$ at an infinite distance from the sphere.

In seeking a solution to the boundary value problem for v posed above for general depths h of the sphere centre below the surfactant layer, it is advantageous to work with bispherical coordinates (ξ, η) which are related to the cylindrical polar coordinates (ρ, z) by the relations

$$\rho = \frac{c \sin \eta}{\cosh \xi - \cos \eta}, \quad z = \frac{c \sinh \xi}{\cosh \xi - \cos \eta}, \qquad [2.6]$$

where c is a constant which has the dimension of length. The plane $z = 0$ corresponds to $\xi = 0$ and the sphere is defined by $\xi = \alpha$, where $a = c \operatorname{cosech} \alpha$ and $h = c \coth \alpha$. Accordingly, for a given choice of a and h, the parameters c and α are uniquely determined. The part of the fluid infinitely far away from the sphere corresponds to $\xi, \eta \to 0$.

The general form of solution to [2.3] in essentially separated bispherical variables ξ, η was worked out by Jeffrey (1915). The appropriate form of solution for our purposes is therefore

$$v = \Omega c(\cosh \xi - \cos \eta)^{1/2} \sum_{n=1}^{\infty} \{A_n \cosh(n + \tfrac{1}{2})\xi + B_n \sinh(n + \tfrac{1}{2})\xi\} P_n{}^1(\cos \eta) \qquad [2.7]$$

where $P_n{}^1(\cos \eta)$ is the associated Legendre function of the first kind and with order n and degree unity. By making use of the expansion

$$(\cosh \xi - \cos \eta)^{-1/2} = \sqrt{2} \sum_{n=0}^{\infty} e^{-(n+1/2)|\xi|} P_n(\cos \eta) \qquad [2.8]$$

given for instance by Morse & Feshbach (1956), it is easy to show that the condition [2.4] on the sphere is satisfied if

$$B_n + k_n A_n = 2\sqrt{2}(k_n - 1), \quad (n \geq 1) \qquad [2.9]$$

where $k_n = \coth(n + \tfrac{1}{2})\alpha$.

With the velocity given by [2.7], it can at once be shown that

$$\frac{\partial v}{\partial z} = -\tfrac{1}{2}\Omega(\cosh \xi - \cos \eta)^{1/2}\sum_{n=1}^{\infty}\{[(n+2)A_{n+1} - (2n+1)A_n + (n-1)A_{n-1}]\sinh(n+\tfrac{1}{2})\xi$$
$$+ [(n+2)B_{n+1} - (2n+1)B_n + (n-1)B_{n-1}]\cosh(n+\tfrac{1}{2})\xi\}P_n^{1}(\cos \eta). \qquad [2.10]$$

Accordingly the surfactant boundary condition [2.5] is satisfied if

$$(n+2)(\bar{\lambda}C_{n+1} + 2B_{n+1}) - (2n+1)(\bar{\lambda}C_n + 2B_n) + (n-1)(\bar{\lambda}C_{n-1} + 2B_{n-1}) = 0, \quad (n \geq 1) \qquad [2.11]$$

where $\bar{\lambda} = \lambda/c$ and

$$C_n = (n+2)A_{n+1} - (2n+1)A_n + (n-1)A_{n-1} \quad (n \geq 1). \qquad [2.12]$$

It therefore follows that for all values of $n \geq 1$,

$$\bar{\lambda}C_n + 2B_n = \text{constant}. \qquad [2.13]$$

To determine this constant, we note that

$$(v)_{z=0} = (v)_{\xi=0} = \Omega c(1-\cos \eta)^{1/2}\sum_{n=1}^{\infty} A_n P_n^{1}(\cos \eta),$$

and since

$$\sum_{n=1}^{\infty} P_n^{1}(\cos \eta) = 2\sqrt{2}\sin \eta(1-\cos \eta)^{-3/2},$$

it follows that if $A_n \to$ a non-zero constant as $n \to \infty$, then $(v)_{\xi=0} = 0(\eta^{-1})$ as $\eta \to 0$. Consequently we require that $A_n \to 0$ as $n \to \infty$ in order to eliminate this singularity. This means that B_n, $C_n \to 0$ as $n \to \infty$ and the constant appearing in [2.13] is zero. Combining [2.9] and [2.13] we obtain

$$\bar{\lambda}\{(n+2)A_{n+1} - (2n+1)A_n + (n-1)A_{n-1}\} - 2k_n A_n = 4\sqrt{2}(1-k_n),$$
$$(n \geq 1). \qquad [2.14]$$

From [2.14], all coefficients A_n can be determined once A_1 is known. To find this coefficient, the difference equation may be solved in the following way. We let $\{T_n\}$ denote the solution of [2.14] with $T_1 = 0$, and $\{U_n\}$ denote the solution of the homogeneous difference equation obtained by setting the r.h.s. of [2.14] equal to zero and setting $U_1 = 1$. The complete solution of [2.14] will then be given by

$$A_n = T_n + A_1 U_n \quad (n \geq 1). \qquad [2.15]$$

Thus since $A_n \to 0$ as $n \to \infty$,

$$A_1 = -\lim_{n\to\infty}\frac{T_n}{U_n}. \qquad [2.16]$$

In our numerical calculations, we used as our criterion for the truncation of the sequences $\{U_n\}$ and $\{T_n\}$ the condition that the difference between the values taken by T_n/U_n in two successive evaluations must be less than 10^{-15}. This ensures that there is a set of values of $\{A_n\}$ known to a high degree of accuracy which is essential for calculating the torque acting on the sphere.

Although for general values of λ, the difference equation [2.14] has to be solved numerically, the solution can be readily found in closed form for the limiting cases when $\lambda \to 0$ and $\lambda \to \infty$. When $\lambda \to 0$, the solution is

$$A_n = 2\sqrt{2}\{1 - \tanh(n + \tfrac{1}{2})\alpha\}, \quad B_n = 0 \quad (n \geq 1). \qquad [2.17]$$

In this limit the boundary condition on $z = 0$ reduces to $\partial v/\partial z = 0$ which is the appropriate boundary condition at a simple stress free surface. This solution is also identical to that for two equal sized spheres rotating with equal angular velocities in an infinite fluid. When $\lambda \to \infty$, the solution is then

$$A_n = 0, \quad B_n = 2\sqrt{2}\{\coth(n + \tfrac{1}{2})\alpha - 1\}, \quad (n \geq 1). \qquad [2.18]$$

Now the boundary condition on $z = 0$ reduces to $\partial^2 v/\partial z^2 = 0$, which, by virtue of [2.3] together with the condition of zero velocity at infinity and at the origin, leads to the equivalent condition that v vanishes on $z = 0$. Consequently this solution is identical to that for a sphere rotating in a semi-infinite fluid bounded by a rigid plane wall. The solutions to both of these limiting problems were first given by Jeffery (1915).

3. THE TORQUE ACTING ON THE SPHERE

As the sphere rotates, the torque which the fluid exerts on the sphere to resist its motion is $\mathbf{T} = -T\mathbf{k}$ where

$$T = 2\pi\mu \int_\gamma \rho^3 \left[\frac{\partial}{\partial n}\left(\frac{v}{\rho}\right)\right] \mathrm{d}s, \qquad [3.1]$$

where the path of integration γ is a meridional section of the sphere and $\partial/\partial n$ denotes differentiation along the direction of the normal to the sphere drawn into the fluid. On substituting for v from [2.7] and expressing the other terms in bispherical coordinates, we obtain the expression

$$T = 2\pi\mu\Omega c^3 \left\{ \tfrac{3}{2} \sinh\alpha \sum_{n=1}^{\infty} W_n(\alpha) \int_0^\pi \frac{\sin^2\eta\, P_n{}^1(\cos\eta)}{(\cosh\alpha - \cos\eta)^{5/2}} \mathrm{d}\eta \right.$$
$$\left. + \sum_{n=1}^{\infty} W_n'(\alpha) \int_0^\pi \frac{\sin^2\eta P_n{}^1(\cos\eta)}{(\cosh\alpha - \cos\eta)^{3/2}} \mathrm{d}\eta \right\} \qquad [3.2]$$

where $W_n(\alpha) = A_n \cosh(n + \tfrac{1}{2})\alpha + B_n \sinh(n + \tfrac{1}{2})\alpha$. By using the relations

$$\frac{\sin\eta}{(\cosh\alpha - \cos\eta)^{3/2}} = 2\sqrt{2} \sum_{n=1}^{\infty} \mathrm{e}^{-(n+1/2)\alpha} P_n{}^1(\cos\eta)$$
$$\frac{\sin\eta}{(\cosh\alpha - \cos\eta)^{5/2}} = \frac{2\sqrt{2}}{3\sinh\alpha} \sum_{n=1}^{\infty} (2n+1)\mathrm{e}^{-(n+1/2)\alpha} P_n{}^1(\cos\eta),$$

together with the orthogonality relation for the Legendre functions, it follows that

$$T = 4\sqrt{2}\pi\mu\Omega a^3 \sinh^3\alpha \sum_{n=1}^{\infty} n(n+1)[A_n + B_n]. \qquad [3.3]$$

On defining a dimensionless torque coefficient τ as $T/8\pi\mu\Omega a^3$ and substituting for B_n we find that

$$\tau = \frac{1}{\sqrt{2}} \sinh^3 \alpha \sum_{n=1}^{\infty} n(n+1)(2\sqrt{2} - A_n)[\coth(n+\tfrac{1}{2})\alpha - 1]. \qquad [3.4]$$

A further simplification can be obtained by noting that

$$\sum_{n=1}^{\infty} n(n+1)[\coth(n+\tfrac{1}{2})\alpha - 1] = \tfrac{1}{2}\sum_{n=1}^{\infty} \operatorname{cosech}^3 n\alpha. \qquad [3.5]$$

Consequently [3.4] can be written as

$$\tau = \sinh^3 \alpha \sum_{n=1}^{\infty} \operatorname{cosech}^3 n\alpha - \frac{1}{\sqrt{2}} \sinh^3 \alpha \sum_{n=1}^{\infty} n(n+1) A_n [\coth(n+\tfrac{1}{2})\alpha - 1]. \qquad [3.6]$$

For the case when $\lambda \to 0$, [3.6] reduces to

$$\tau = \sinh^3\alpha \sum_{n=1}^{\infty} (-1)^{n+1} \operatorname{cosech}^3 n\alpha. \qquad [3.7]$$

When $\alpha \to \infty$, the sphere is then at an infinite distance from the free surface and we obtain the expected limit $\tau \to 1$. As $\alpha \to 0$, the sphere approaches tangency with the free surface and we now obtain

$$\tau \to \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} = \tfrac{3}{4}\zeta(3) = 0.901543$$

where $\zeta(x)$ denotes the Riemann zeta function. For the case when $\lambda \to \infty$, we find that in this limit,

$$\tau = \sinh^3\alpha \sum_{n=1}^{\infty} \operatorname{cosech}^3 n\alpha. \qquad [3.8]$$

When $\alpha \to \infty$, we again see that $\tau \to 1$, but when $\alpha \to 0$,

$$\tau \to \zeta(3) = 1.20206.$$

4. DEEPLY SUBMERGED SPHERE

When $h \gg a$, a simple solution, valid for all values of λ, can be obtained by the method of matched asymptotic expansions. Near the sphere, in the "inner region", the solution for the velocity v is essentially unaffected by the presence of the surfactant layer at $z = 0$. Thus v must satisfy [2.3] and [2.4] together with the vanishing condition at infinity. Hence, to leading order,

$$v \sim \frac{\Omega\rho}{[(z-h)^2 + \rho^2]^{3/2}}. \qquad [4.1]$$

However in the "outer region", the effect of the sphere on the solution is that of a point singularity. Now the velocity V must satisfy [2.3], [2.5], the vanishing condition at infinity and in addition, a singularity condition derived from [4.1]. Since h is the appropriate length scale for the outer region, we write

$$\hat{\rho} = \rho/h, \quad \hat{z} = z/h, \tag{4.2}$$

and [2.5] and [4.1] yield

$$\frac{\partial V}{\partial \hat{z}} = \frac{\lambda}{h}\frac{\partial^2 V}{\partial \hat{z}^2} \quad (\hat{z}=0), \tag{4.3}$$

$$V \sim \frac{\Omega\hat{\rho}}{h^2[(\hat{z}-1)^2+\hat{\rho}^2]^{3/2}} + \text{regular function}, \tag{4.4}$$

as $(\rho, z)\rightarrow(0,1)$. It can then be readily shown that for [2.3] and the condition at infinity to be satisfied, V must to leading order be given by

$$V \sim \frac{\Omega\hat{\rho}}{h^2}\{[(\hat{z}-1)^2+\hat{\rho}^2]^{-3/2} + [(\hat{z}+1)^2+\hat{\rho}^2]^{-3/2}\} - \frac{2\lambda\Omega}{h^2}\int_0^\infty \frac{k^2 e^{-k(1+\hat{z})}}{\lambda k + h} J_1(k\hat{\rho})\mathrm{d}k. \tag{4.5}$$

In particular, on the surface $\hat{z}=0$, the velocity is

$$V \sim \frac{2\Omega\hat{\rho}}{h^2}(1+\hat{\rho}^2)^{-3/2} - \frac{2\lambda\Omega}{h^2}\int_0^\infty \frac{k^2e^{-k}}{\lambda k + h} J_1(k\hat{\rho})\mathrm{d}k = \frac{2\Omega\rho}{(h^2+\rho^2)^{3/2}} - 2\lambda\Omega\int_0^\infty \frac{s^2 e^{-sh}}{\lambda s + 1} J_1(s\rho)\mathrm{d}s \tag{4.6}$$

where $s = k/h$. After using Watson's lemma and the fact that $J_1(x)\sim\frac{1}{2}x$ as $x\rightarrow 0$, [4.6] gives

$$V \sim \frac{2\Omega\rho}{(h^2+\rho^2)^{3/2}} - 2\lambda\Omega\left\{\frac{3\rho h}{(h^2+\rho^2)^{5/2}} + \mathrm{O}(\lambda)\right\} \sim \frac{2\Omega\rho}{h^3}\left(1-\frac{3\lambda}{h}\right), \tag{4.7}$$

when $\rho/h \ll 1$. In addition, [4.5] can be used to improve the "inner solution" given by [4.1]. Expression [4.5] shows that the far field behaviour of v must be of the form

$$\begin{aligned} v &\sim \Omega\rho\{[(z-h)^2+\rho^2]^{-3/2} + [(z+h)^2+\rho^2]^{-3/2}\} \\ &\quad -2\lambda\Omega\int_0^\infty \frac{s^2 e^{-s(h+z)}}{\lambda s+1} J_1(s\rho)\mathrm{d}s \\ &\sim \frac{\Omega\rho}{[(z-h)^2+\rho^2]^{3/2}} + \frac{\Omega\rho}{8h^3} - \lambda\Omega\rho\int_0^\infty s^3 e^{-2sh}\mathrm{d}s \\ &= \frac{\Omega\rho}{[(z-h)^2+\rho^2]^{3/2}} + \frac{\Omega\rho}{8h^3}\left(1-\frac{3\lambda}{h}\right), \end{aligned} \tag{4.8}$$

and in order that condition [2.4] is not violated, the inner region solution must be given by

$$v \sim \frac{\Omega\rho}{[(z-h)^2+\rho^2]^{3/2}} + \frac{\Omega}{8h^3}\left(1-\frac{3\lambda}{h}\right)\left[\rho - \frac{\rho}{[(z-h)^2+\rho^2]^{3/2}}\right]. \tag{4.9}$$

Expressions [4.7] and [4.9] apply when $\lambda \ll h$. For the more practically useful situation when $\lambda \gg h$, a more suitable form for V, obtained by rearranging [4.5], is given by

$$V \sim \frac{\Omega\hat{\rho}}{h^2}\{[(\hat{z}-1)^2+\hat{\rho}^2]^{-3/2} - [(\hat{z}+1)^2+\hat{\rho}^2]^{-3/2}\} + \frac{2\Omega}{h}\int_0^\infty \frac{k e^{-k(\hat{z}+1)}}{\lambda k + h} J_1(k\hat{\rho})\mathrm{d}k. \tag{4.10}$$

The first two terms are the solution in the limit $\lambda \to \infty$, where the condition $\partial^2 V/\partial \hat{z}^2 = 0$ on $\hat{z} = 0$ with V bounded for all $\hat{\rho}$ implies that $V = 0$ on $\hat{z} = 0$. In particular the velocity on the surface $\hat{z} = 0$ is given by

$$V \sim 2\Omega \int_0^\infty \frac{s\,\mathrm{e}^{-sh}}{\lambda s + 1} J_1(s\rho)\,\mathrm{d}s$$

which can be shown to be

$$\sim \frac{2\Omega}{\lambda} \int_0^\infty \mathrm{e}^{-sh} J_1(s\rho)\,\mathrm{d}s = \frac{2\Omega\rho}{\lambda(h^2+\rho^2)}\left[1 + \frac{h}{(h^2+\rho^2)^{1/2}}\right]^{-1}. \qquad [4.11]$$

The far field behaviour of the inner solution v must be of the form

$$v \sim \frac{\Omega\rho}{[(z-h)^2+\rho^2]^{3/2}} - \frac{\Omega\rho}{8h^3} + \Omega\rho \int_0^\infty \frac{s^2 \mathrm{e}^{-2sh}}{\lambda s + 1}\,\mathrm{d}s$$

$$\sim \frac{\Omega\rho}{[(z-h)^2+\rho^2]^{3/2}} - \frac{\Omega\rho}{8h^3}\left[1 - \frac{2h}{\lambda} + 0\left(\frac{h^2}{\lambda^2}\right)\right]. \qquad [4.12]$$

It therefore follows that

$$v \sim \frac{\Omega\rho}{[(z-h)^2+\rho^2]^{3/2}} - \frac{\Omega}{8h^3}\left(1 - \frac{2h}{\lambda}\right)\left[\rho - \frac{\rho}{[(z-h)^2+\rho^2]^{3/2}}\right] \qquad [4.13]$$

The torque exerted on the sphere by the fluid is given by

$$T = -2\pi\mu a^4 \int_0^\pi \left[\frac{\partial}{\partial R}\left(\frac{v}{\rho}\right)\right]_{R=1} \sin^3\theta\,\mathrm{d}\theta, \qquad [4.14]$$

where μ is the viscosity and $\rho = R\sin\theta$, $z - h = R\cos\theta$. On substituting [4.9], it readily follows that

$$T = 6\pi\mu\Omega a^3\left\{1 - \frac{a^3}{8h^3} + \frac{3\lambda a^3}{8h^4}\right\}\int_0^\pi \sin^3\theta\,\mathrm{d}\theta$$

$$= 8\pi\mu\Omega a^3\left\{1 - \frac{a^3}{8h^3}\left(1 - \frac{3\lambda}{h}\right)\right\}. \qquad [4.15]$$

It will be noticed that [4.7] and [4.15] are valid for all values of λ such that $\lambda/h \ll 1$, so that λ itself need not be small if the depth of the sphere is large. The correction term in [4.15] is closely related to the velocity distribution of [4.7]. The corresponding value of the torque when $\lambda \gg h \gg a$ is, on using [4.13], given by

$$T = 8\pi\mu\Omega a^3\left\{1 + \frac{a^3}{8h^3}\left(1 - \frac{2h}{\lambda}\right)\right\}. \qquad [4.16]$$

5. SUBSTRATE OF FINITE DEPTH

In this section, we consider the effect on the velocity and torque when the substrate fluid has a finite depth H, where $H/a \gg 1$. We shall also suppose that $h/a \gg 1$ and $(H-h)/a \gg 1$. In constructing the solution for the outer region velocity V, we need to modify the Green's function so as to take into account the condition that the velocity vanishes on the rigid boundary $z = H$, which, in terms of the stretched variables defined by [4.2], is given by $\hat{z} = \gamma$ where $\gamma = H/h$. Accordingly the asymptotic form of the outer region velocity is given by

$$V \sim \frac{\Omega\hat{\rho}}{h^2}\sum_{m=-\infty}^{\infty}\{[(\hat{z}-2m\gamma-1)^2+\hat{\rho}^2]^{-3/2}-[(\hat{z}-2m\gamma+1)^2+\hat{\rho}^2]^{-3/2}\}$$
$$+\frac{2\Omega}{h^2}\int_0^\infty \frac{k\sinh k(\gamma-1)\sinh k(\gamma-\hat{z})J_1(k\hat{\rho})dk}{[\cosh k\gamma+(\lambda k/h)\sinh k\gamma]\sinh k\gamma}. \quad [5.1]$$

This form of expression is most suitable when $\lambda \gg h$ and, as in [4.13], the integral vanishes in the limit $\lambda \to \infty$, so the torque is therefore given asymptotically by

$$T \sim 8\pi\mu\Omega a^3\left\{1-\frac{\zeta(3)a^3}{4H^3}+\frac{1}{8}\sum_{m=-\infty}^{\infty}\frac{a^3}{|mH-h|^3}-\frac{a}{\lambda}\int_0^\infty s\left[\frac{\sinh s(H-h)}{\sinh sH}\right]^2 ds\right\}. \quad [5.2]$$

The corresponding expressions for $\lambda \ll h$, are given by

$$V \sim \frac{\Omega\hat{\rho}}{h^2}\sum_{m=-\infty}^{\infty}(-1)^m\{[(\hat{z}-2m\gamma-1)^2+\rho^2]^{-3/2}+[(\hat{z}-2m\gamma+1)^2+\hat{\rho}]^{-3/2}\}$$
$$-\frac{2\lambda\Omega a^2}{h^3}\int_0^\infty \frac{k^2\sinh k(\gamma-1)\sinh k(\gamma-\hat{z})J_1(k\hat{\rho})}{[\cosh k\gamma+(\lambda k/h)\sinh k\gamma]\cosh k\gamma}dk \quad [5.3]$$

and

$$T \sim 8\pi\mu\Omega a^3\left\{1+\frac{3\zeta(3)a^3}{16H^3}-\frac{1}{8}\sum_{m=-\infty}^{\infty}\frac{(-1)^m a^3}{|mH-h|^3}+\frac{\lambda}{a}\int_0^\infty s^3\left[\frac{\sinh s(H-h)}{\cosh sH}\right]^2 ds\right\}. \quad [5.4]$$

If $H \gg h$, then [5.2] becomes

$$T \sim 8\pi\mu\Omega a^3\left\{1+\frac{a^3}{8h^3}\left(1-\frac{2h}{\lambda}\right)\right\},$$

which is the same as [4.16]. However [5.4] becomes

$$T \sim 8\pi\mu\Omega a^3\left\{1-\frac{a^3}{8h^3}\left(1-\frac{3\lambda}{h}\right)+\frac{3\zeta(3)a^3}{8H^3}\right\}$$

which reduces to [4.15] in the limit when $H/a \to \infty$.

6. RESULTS OF THE NUMERICAL WORK

We have solved the difference equation [2.14] for a large set of values of $\bar{\lambda}$ in the range $0.1 \leq \bar{\lambda} \leq 100$. The method employed has been described in detail in section 2 and it proved to be a particularly simple method requiring very little computer time in order to achieve very great accuracy as is needed in order to calculate the torque and surface velocity distribution. In our calculations, the criterion which we used to truncate the solution sequence $\{A_n\}$ was the stabilization of the ratio T_n/U_n so that in two successive evaluations with increasing n, this ratio did not change by more than 10^{-15}. The numerical method allows the solution of [2.14] to be found with facility for all values of the sphere depth parameter. We found that the number of effectively non-zero terms in the sequence $\{A_n\}$ for any given α increases with $\bar{\lambda}$, and for any given $\bar{\lambda}$, increases as α decreases. For each value of $\bar{\lambda}$ considered, we carried out our

Table 1.

α	h/a	$\bar\lambda=0.2$ τ	τ/τ_0	$\bar\lambda=0.4$ τ	τ/τ_0	$\bar\lambda=0.6$ τ	τ/τ_0
2.0	3.7622	0.9986	1.0010	0.9992	1.0016	1.0000	1.0020
1.0	1.5431	0.9786	1.0115	0.9862	1.0193	0.9917	1.0250
0.5	1.1276	0.9401	1.0162	0.9524	1.0296	0.9628	1.0409
0.2	1.0201	0.9134	1.0086	0.9207	1.0166	0.9269	1.0235
0.1	1.0050	0.9064	1.0042	0.9098	1.0080	0.9110	1.0093
		$\bar\lambda=2.0$		$\bar\lambda=4.0$		$\bar\lambda=6.0$	
2.0	3.7622	1.0009	1.0033	1.0015	1.0038	1.0017	1.0041
1.0	1.5431	1.0102	1.0441	1.0194	1.0536	1.0236	1.0580
0.5	1.1276	1.0072	1.0888	1.0362	1.1201	1.0516	1.1369
0.2	1.0201	0.9653	1.0658	1.0015	1.1058	1.0264	1.1333
0.1	1.0050	0.9375	1.0387	0.9642	1.0683	0.9853	1.0916
		$\bar\lambda=10.0$		$\bar\lambda=40.0$		$\bar\lambda=100.0$	
2.0	3.7622	1.0019	1.0043	1.0022	1.0046	1.0023	1.0047
1.0	1.5431	1.0277	1.0622	1.0334	1.0681	1.0348	1.0695
0.5	1.1276	1.0680	1.1566	1.0939	1.1826	1.1007	1.1900
0.2	1.0201	1.0586	1.1689	1.1280	1.2455	1.1515	1.2715
0.1	1.0050	1.0169	1.1266	1.1082	1.2279	1.1500	1.2742

calculations for values of α in the range $5.0>\alpha>0.1$. The extreme values of this range correspond to $h/a=74.2099$ and 1.0050 respectively, giving a very broad spectrum of values of h/a. The largest number of terms of $\{A_n\}$ which had to be calculated was 709 in order to meet the stated truncation criterion. This was for the case $\bar\lambda=100$, $\alpha=0.1$, while on the other hand, for $\bar\lambda=0.1$, no more than 13 terms had to be calculated for any value of α. For any value of $\bar\lambda$, it was found that the torque τ acting on the sphere is insenstiive to the presence of the surfactant until $\alpha\approx2.0$, corresponding to $h/a\approx3.76$. If α is decreased to 0.1, then the torque decreases monotonically from 1.0, its value when $h/a\to\infty$, provided that $\bar\lambda<\bar\lambda_0$ where $\bar\lambda_0\approx0.775$. For values of $\bar\lambda$ exceeding this critical value, decreasing α results in an increase in τ from 1.0 to a maximum and then a decrease to its least value as $\alpha\to0$. The value of α at which the maximum occurs depends on $\bar\lambda$ in such a way that it decreases as $\bar\lambda$ increases. This behaviour is illustrated in table 1 where we have listed both the torque τ and the ratio τ/τ_0, where τ_0 denotes the value of the torque for that particular sphere depth when $\kappa=0$ and the surfactant layer degenerates into a simple stress free surface. This latter quantity provides a good measure of the strength of the surfactant in influencing the motion of the substrate. Clearly for any given value of α and $\bar\lambda$, we have $1<\tau/\tau_0<\tau_\infty/\tau_0$, where τ_∞ denotes the value of τ for that choice of α when $\bar\lambda=\infty$. In table 2, we have displayed the values of τ_0, τ_∞ and τ_∞/τ_0 for various values of α, using the formulae given by [3.7] and [3.8].

In figure 1, we have plotted the graphs of τ/τ_0 as a function of h/a for $\bar\lambda=0.2$, 0.4, 0.6, 4.0, 6.0 and $\bar\lambda=\infty$.

The surface velocity is given by [2.7] when we set $\xi=0$. We therefore have the expression

$$v/\Omega a=\sinh\alpha(1-\cos\eta)^{1/2}\sum_{n=1}^{\infty}A_nP_n^{\,1}(\cos\eta). \qquad [6.1]$$

Table 2.

α	h/a	τ_0	τ_∞	τ_∞/τ_0
2.0	3.7622	0.9977	1.0024	1.0047
1.0	1.5431	0.9675	1.0357	1.0705
0.5	1.1276	0.9250	1.1056	1.1952
0.2	1.0201	0.9056	1.1709	1.2930
0.1	1.0050	0.9026	1.1910	1.3195
0.0	1.0000	0.9015	1.2021	1.3333

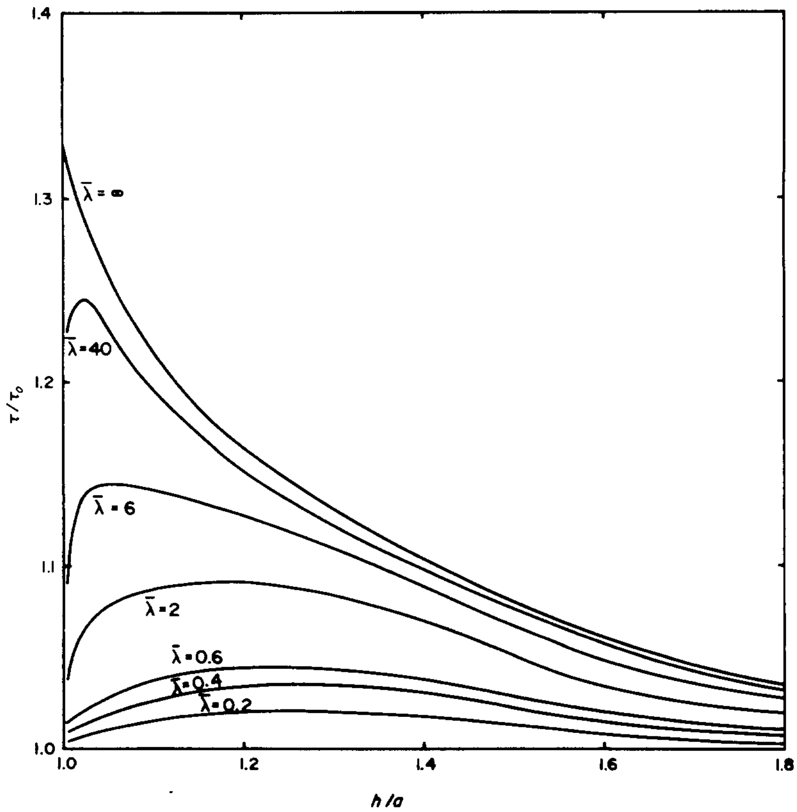

Figure 1. Graphs of τ/τ_0 plotted vs h/a for various fixed values of $\bar{\lambda}$.

The radial distance on the surface can be obtained from [2.6] with $\xi = 0$. This gives

$$\rho/a = \sinh \alpha \cot \tfrac{1}{2} \eta. \qquad [6.2]$$

In table 3 we display values of the dimensionless surface velocity for various values of the parameters $\bar{\lambda}$, α and ρ/a. The values selected for $\bar{\lambda}$ were 0.2, 0.6, 1.0, 2.0, 6.0 and 10.0 and the values of α chosen were 0.5, 0.3 and 0.1 which correspond to values of $h/a = 1.1276$, 1.0453 and 1.0050 respectively. In figure 2, we have plotted graphs of $v/\Omega a$ vs ρ/a for $\alpha = 0.5$.

Measurement of the surface velocity is clearly an alternative experimental procedure to the measurement of the torque. When the parameter λ is small, the torque is then less sensitive to the presence of the surfactant than in the case when λ is large. However when λ is small, the variation in the surface velocity is larger than when λ is large. Thus we might expect that either measurement technique has its use depending on the size of the surface shear viscosity of the adsorbed film.

In conclusion, we would like to point out that the problem studied in this paper is one which has an exact mathematical solution from which those physical quantities such as torque and surface velocity can be calculated as accurately as we please for any depth of the sphere below the surface of the adsorbed film. The choice of a sphere as the rotating body is also advantageous in that it is a shape which can be manufactured quite easily with great precision, thus both the geometrical parameters of sphere radius and depth can be accurately measured and controlled. Only these two geometrical parameters are required for evaluating the exact mathematical solution for a given value of λ.

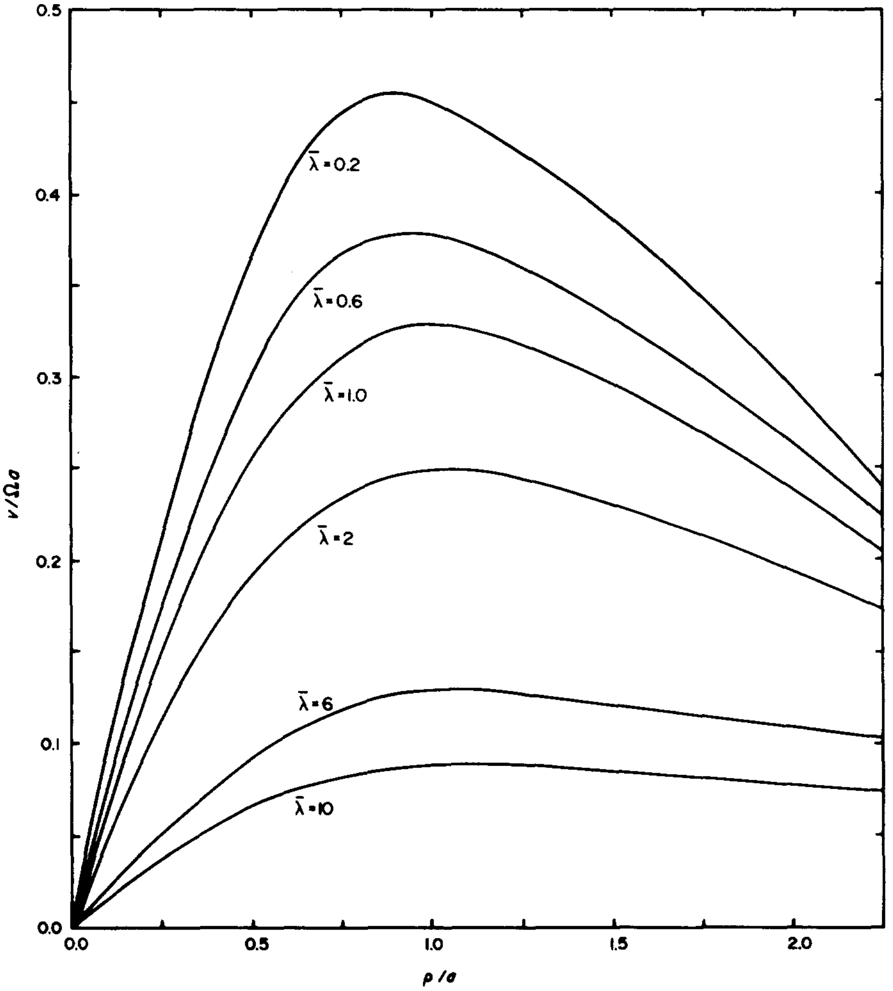

Figure 2. Graphs of $v/\Omega a$ plotted vs ρ/a for $\alpha = 0.5$, corresponding to $h/a = 1.1276$.

Table 3.

$\alpha = 0.5$, $h/a = 1.276$ $\cos\eta$	ρ/a	$\bar\lambda = 0.2$	$\bar\lambda = 0.6$	$v/\Omega a$ $\bar\lambda = 1.0$	$\bar\lambda = 6.0$	$\bar\lambda = 10.0$
−1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
−0.8	0.1737	0.1512	0.1257	0.1094	0.0399	0.0268
−0.6	0.2605	£.2205	0.1827	0.1560	0.0580	0.0390
−0.4	0.3411	0.2785	0.2300	0.1980	0.0730	0.0491
−0.2	0.4255	0.3309	0.2727	0.2326	0.0868	0.0584
0.0	0.5211	0.3790	0.3119	0.2667	0.0998	0.0673
0.2	0.6382	0.4209	0.3469	0.2975	0.1121	0.0758
0.4	0.7960	0.4501	0.3733	0.3203	0.1232	0.0836
0.6	1.0422	0.4488	0.3786	0.3292	0.1308	0.0893
0.8	1.5633	0.3644	0.3208	0.2858	0.1249	0.0869
0.9	2.2714	0.2443	0.2255	0.2085	0.1031	0.0737
$\alpha = 0.3$, $h/a = 1.0453$						
−1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
−0.8	0.1015	0.1004	0.0955	0.0896	0.0467	0.0337
−0.6	0.1523	0.1499	0.1419	0.1327	0.0688	0.0495
−0.4	0.1994	0.1950	0.1834	0.1710	0.0879	0.0633
−0.2	0.2486	0.2409	0.2249	0.2088	0.1064	0.0765
0.0	0.3045	0.2905	0.2688	0.2485	0.1254	0.0901
0.2	0.3730	0.3464	0.3173	0.2919	0.1456	0.1046
0.4	0.4652	0.4110	0.3723	0.3405	0.1680	0.1206
0.6	0.6090	0.4823	0.4325	0.3935	0.1926	0.1383
0.8	0.9136	0.5236	0.4704	0.4288	0.2135	0.1543
0.9	1.3274	0.4524	0.4158	0.3849	0.2043	0.1499
$\alpha = 0.1$, $h/a = 1.005$						
−1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
−0.8	0.0334	0.0437	0.0334	0.0334	0.0308	0.0277
−0.6	0.0501	0.0551	0.0502	0.0500	0.0461	0.0413
−0.4	0.0656	0.0700	0.0657	0.0655	0.0600	0.0536
−0.2	0.0818	0.0891	0.0818	0.0817	0.0743	0.0662
0.0	0.1002	0.1074	0.1002	1.0000	0.0902	0.0801
0.2	0.1227	0.1259	0.1226	1.2236	1.0919	0.0965
0.4	0.1530	0.1527	0.1528	1.5233	1.3371	1.1736
0.6	0.2003	0.2005	0.1997	1.9847	1.6953	1.4731
0.8	0.3005	0.3084	0.2954	2.9150	2.3520	2.0070
0.9	0.4366	0.4239	0.4096	3.9917	3.0264	2.5387

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